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# $U(2, 2)$ Symmetry as a Common Basis for Quantum Theory and Geometrodynamics

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We formulated some criticisms of the Dirac equation and its Clifford-algebraic philosophy; in particular, we show that, within a general-relativistic context, they seem to contain hidden action-at-distance concepts. We suggest a new model based on the four-component Klein–Gordon equation locally invariant under the $U(2,2)$ gauge group. The usual Dirac equation is then obtained as a certain approximation. The geometrodynamical sector shows reasonable correspondence with general relativity.

## 1. INTRODUCTION. OBJECTIONS AGAINST DIRAC THEORY

The historical incompatibility between the Hilbert space formalism of quantum mechanics and relativity theory has been one of the greatest challenges of twentieth century physics. At the same time, either of these disciplines taken in itself has its own problems, like measurement in quantum mechanics and singularities in gravitation. As both theories contain a profound physical truth, the idea appeared that somewhere at the interface of quanta and relativity a new theory might appear which, including their successes, could also solve their paradoxes and unify them into a compatible whole. The essential nonlinearity of generally covariant theories could perhaps overcome quantum mechanical measurement paradoxes, whereas quantum effects could (as a matter of fact, they do) modify our understanding of gravitational singularities. There is a philosophical paradigm due to Finkelstein, Penrose, and Weizsäcker (Castell *et al.*, 1975, Penrose and Rindler, 1984) according to which the common roots of quanta and relativity are based on the geometry of two-component spinors, describing the most elementary physical entities.

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The superposition principle, wave–particle dualism, the four-dimensionality of space-time and its normal-hyperbolic signature, perhaps even the arrow of time, seem to be somehow unified just on the level of the two-component spinor as a $\mathbb{C}$-linear shell of the elementary yes–no alternative $Z_2$ of irreducible measurements. Within this philosophy, the Dirac bispinor appears as a secondary construction—an important element of the tower of algebraic structures over $\mathbb{C}^2$. Historically, bispinors appeared in a different way, in Dirac's attempts at unifying quantum mechanics and special relativity and overcoming the known failure of earlier efforts based on the Klein–Gordon equation.

In this way, Clifford algebras have been an important mathematical tool of physics, together with the linearization idea of the d'Alembert operator, represented as the second power of the first-order Dirac operator, $g^{\mu\nu}\partial_\mu\partial_\nu I = -(i\gamma^\mu\partial_\mu)^2$; classically, $g^{\mu\nu}p_\mu p_\nu I = (\gamma^\mu p_\mu)^2$. Although it turned out rather quickly, that the Dirac equation also could not work consistently as a one-particle quantum mechanical equation, after the fermionic second quantization based an anticommutators, it became an indispensable foundation of the theory of elementary particles and fundamental interactions. In this way, Clifford algebras and the corresponding linearization idea have been commonly recognized as the fundamental paradigm of physics.

In our opinion, the critical analysis of the interface between quanta and relativity theory (both special and general) should start just here, from certain objections against the Dirac theory. These objections seem to us particularly convincing and natural within the general-relativistic context. In spite of the amazing theoretical success of Dirac theory, we dare to raise the following reproaches against it (Sławianowski 1996, 1997):

(i) Bad transformation properties under pseudo-unitary group $U(2, 2)$ and its quotient conformal group $C(1, 3)$, although the internal metric $(++--)$ is an inherent element of Dirac theory.

(ii) Mysterious structure of the Dirac Lagrangian, which is linear in a quantity of the bosonic current structure, as if implied by the $U(2, 2)$ symmetry on some hidden, boson-like level.

(iii) The worst thing consists in hidden action-at-distance elements and certain global rigidity, incompatible with the local paradigm of gauge theories. In special relativity, the representation of Dirac matrices is physically irrelevant. In general relativity, this pattern is followed, although there are no sufficient reasons for that. It is more natural to allow the representation to change smoothly and dynamically over the space-time manifold, and this should lead in principle to observable effects. How are remote quasars to know the representation we decided to use?

(iv) In general relativity, the Clifford paradigm loses its conceptual coherence and convincing power. Classically, it is still true, of course, that $(\gamma^\mu p_\mu)^2 = g^{\mu\nu}\, p_\mu p_\nu I$, but the invariant d'Alembert operator $g^{\mu\nu}iD_\mu iD_\nu I$ built of the spinor covariant derivative $D_\mu$ no longer equals $(i\gamma^\mu D_\mu)^2$. In Riemannian space-time, the latter expression contracts to the form $fI$, but $f$ also contains some term proportional to the curvature scalar. The situation becomes even worse in Riemann–Cartan space-times, which seem to provide a more appropriate framework for spinor theory. Namely, there is no contraction to a $c$-number term, and $(i\gamma^\mu D_\mu)^2$ involves nontrivial matrix terms. Clifford aesthetics disappears, and nonphysical globality becomes a price paid for nothing.

(v) Spinor theory must use the tetrad as one of the gravitational potentials. There is no other "true" gauge theory using reference frames as dynamical variables. Here it is necessary because the universal covering group of $SL(n, \mathbb{R})$ is no longer linear.

In the sequel we aim to show that these objections are answered if one uses as a fundamental framework the four-component Klein–Gordon equation with the local gauge group $U(2, 2)$. This group and the number of components are noncontingent if one believes in the Finkelstein–Penrose–Weizsäcker paradigm, as we do. Dirac theory is a low-energy limit of our model, and the geometrodynamical sector shows a reasonable correspondence with gravitation theory.

## 2. SECOND-ORDER DYNAMICAL MODEL WITH $U(2, 2)$ LOCAL INVARIANCE

Our dynamical variables are the four-component complex wave amplitude $\psi^r$, the connection form $A^r_{s\mu}$ taking values in $u(4,G)$, i.e., in the Lie algebra of pseudo-unitary group $U(4,G)$, and the normal-hyperbolic field $g_{\mu\nu}$. The hermitian form $G_{\bar{\tau}s}$ of signature $(\underline{+}+--)$ is our absolute element. The Dirac conjugate is defined as $\tilde{\psi}_\tau := \bar{\psi}^s G_{\bar{s}r}$. Of course, $U(4,G) \simeq U(2, 2)$. The $A$-covariant differentiation involves two coupling constants: $g$, referring to the simple part $SU(4,G)$, and electromagnetic $q$, referring to the trace part. The Levi-Civita covariant derivative of world quantities is also used; the corresponding affine connection symbol is $\{^{\lambda}_{\mu\nu}\}$. It is convenient to thin unify $\{\ \}$- and $A$-differentiations into a single operation $\nabla_\mu$. The curvature form of $A$ will be denoted by $F = \nabla A$; of course,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \tag{1}$$

Lagrangians for matter and geometry are given by

$$L_m(\psi; A, g) = \frac{b}{2} g^{\mu\nu}\nabla_\mu\tilde{\psi}\nabla_\nu\psi\sqrt{|g|} - \frac{c}{2}\tilde{\psi}\psi\sqrt{|g|} \tag{2}$$

$$L_{YM}(A, g) = \frac{a}{4}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})\sqrt{|g|} + \frac{a'}{4}\operatorname{Tr} F_{\mu\nu}\operatorname{Tr} F^{\mu\nu}\sqrt{|g|} \tag{3}$$

$$L_{HE}(g) = -dR(g)\sqrt{|g|} + l\sqrt{|g|} \tag{4}$$

where $b$, $c$, $a$, $a'$, $d$, $l$ are constants, and $R(g)$ denotes the curvature scalar assigned to $g$. We did not make $d$, $l$ precise here, but it must be stressed that it is not only admissible, but perhaps suggested by analogy with the Palatini principle, to admit the vanishing values $d = 0$, $l = 0$, retaining, nevertheless, the dynamical character (variation) of $g_{\mu\nu}$ in the total Lagrangian. Nonvanishing $d$, $l$, although acceptable, are probably redundant, just as they would have been in the Palatini principle. Nevertheless, here we do not make their values precise and put the total Lagrangian as $L = L_m + L_{YM} + L_{HE}$. Lagrangians lead to dynamical quantities, like the gauge field momentum, symmetric energy-momentum tensors, and the $U(2, 2)$ current of matter:

$$H^{\mu\nu} = -aF^{\mu\nu}\sqrt{|g|} - a'\mathrm{I}\,\mathrm{Tr}\,F_{\mu\nu}\sqrt{|g|} \tag{5}$$

$$T^{\mu\nu} = T^{\mu\nu}_{m} + T^{\mu\nu}_{YM} \tag{6}$$

$$I_{\mu} = \frac{b}{2}(\psi\nabla_{\mu}\tilde{\psi} - \nabla_{\mu}\psi\tilde{\psi})\sqrt{|g|} \tag{7}$$

The field equations read

$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\psi + \frac{c}{b}\psi = 0 \tag{8}$$

$$\nabla_{\nu}H^{\mu\nu} = gI^{\mu} + \frac{q - g}{4}\,\mathrm{Tr}\,I^{\mu}\mathrm{I} \tag{9}$$

$$d\left(R(g)^{\mu\nu} - \frac{1}{2}R(g)g^{\mu\nu}\right) = -\frac{l}{2}g^{\mu\nu} + \frac{1}{2}T^{\mu\nu} \tag{10}$$

[Recall that $\nabla_{\mu}$ acts on internal indices through the $U(2,2)$ connection $A$, and on world indices as the Levi-Civita connection { }.] To discuss the correspondence with standard theory, we must perform the reduction to $SL(2, \mathbb{C})$ or $GL(2, \mathbb{C})$ injected into $U(4, G)$. It is clear that $u(4, G) = iH(4, G)$, where $H(4, G)$ denotes the space of $G$-hermitian operators on $\mathbb{C}^4$. Let us now fix an arbitrary representation of Dirac matrices $\gamma^A$, where, of course,

$$\gamma^A\gamma^B + \gamma^B\gamma^A = 2\eta^{AB}\mathrm{I}, \qquad [\eta_{AB}] = \mathrm{diag}(1, -1, -1, -1), \qquad \gamma^A \in H(4, G)$$

As usual, the remaining $\mathbb{C}$-basic elements of $L(4,\mathbb{C})$ are fixed as

$$\gamma^5 = -\gamma_5 = -\gamma^0\gamma^1\gamma^2\gamma^3, \qquad {}^{A}\gamma = i\gamma^A\gamma^5 = -i\gamma^5\gamma^A$$

$$\Sigma^{AB} = \frac{1}{4}(\gamma^A\gamma^B - \gamma^B\gamma^A) = -\Sigma^{BA}$$

It is interesting that ${}^{A}\gamma^{B}\gamma + {}^{B}\gamma^{A}\gamma = -2\eta^{AB}\mathrm{I}$. The $\mathbb{R}$-linear shell of Dirac matrices,

$$V := \bigotimes_{A=0}^{3} \mathbb{R}\gamma^{A}$$

is normal-hyperbolic with respect to $A, B, \mapsto \frac{1}{4}Tr(AB)$. Obviously, $u(4, G) = iH(4,G)$ is an $\mathbb{R}$-linear shell of matrices:

$$i\gamma^{A}, \quad i\,{}^{A}\gamma, \quad \Sigma^{AB}, \quad i\gamma^{5}, \quad i\mathrm{I}$$

It is also convenient to use combinations:

$$\tau_A = \frac{1}{2}(\gamma_A + {}_A\gamma), \qquad \chi^A = \frac{1}{2}(\gamma^A - {}^A\gamma)$$

The shift of uppercase indices is meant in the $\eta$-sense. In twistor theory $\tau_A$ are known as generators of translations, whereas $\chi^A$ generate proper conformal boosts.

Let us expand the connection form as follows:

$$A_\mu = \frac{1}{2g}\Gamma^{AB}_{\mu}(\Sigma_{AB} + \frac{1}{4}\eta_{AB}\frac{1}{i}\gamma^5) + e^{A}_{\mu}i\tau_A + f_{A\mu}i\chi^A + A'_\mu i\mathrm{I} \tag{11}$$

where $\Gamma^{AB}_{\mu}$ are subject to the condition

$$\check{\Gamma}^{AB}_{\mu} = \Gamma^{AB}_{\mu} - \frac{1}{2}Q_\mu\eta^{AB} = -\check{\Gamma}^{BA}_{\mu} \tag{12}$$

to be satisfied for a certain one-form $Q_\mu$ known as the Weyl covector. Obviously, Lie algebras spanned by $(\Sigma_{AB})$, $(\Sigma_{AB}, i\gamma^5)$, and $(\Sigma_{AB}, i\gamma^5, i\mathrm{I})$, are, respectively, $sl(2, \mathbb{C}) = so(1, 3)$, Weyl algebra (linear-conformal), and $gl(2, \mathbb{C})$ (Weyl algebra with the electromagnetic gauging).

The bispinor connection is given by

$$\omega_\mu = \frac{1}{2}\Gamma^{AB}_{\mu}(\Sigma_{AB} + \frac{1}{4}\eta_{AB}\frac{1}{i}\gamma^5) + A'_\mu i\mathrm{I} \tag{13}$$

It becomes the usual $sl(2, \mathbb{C}) = so(1, 3)$-ruled spinor connection when we put $Q_\mu = 0$, $A'_\mu = 0$.

The corresponding covariant differentiation of bispinors is given by

$$D_\mu\psi = \partial_\mu\psi + \frac{1}{2}\Gamma^{AB}_{\mu}(\Sigma_{AB} + \frac{1}{4}\eta_{AB}\frac{1}{i}\gamma^5)\psi + qA'_\mu i\psi \tag{14}$$

The automorphism group generated by $(\Sigma_{AB}, i\gamma^5, i\mathrm{I})$ preserves $V$ and acts there as Weyl transformations. Restricting the inhomogeneous transformation law for $A$,

$$(UA)_x = U(x)A_xU(x)^{-1} - dU_xU(x)^{-1} \tag{15}$$

to the mentioned group, we obtain the rule

$$e^{K\prime}_{\mu} = L^{K}_{M}e^{M}_{\mu}, \qquad f_{K\mu}{}' = f_{M\mu}L_{K}^{-1M} \tag{16}$$

$$\Gamma^{K}_{N\mu}{}' = L^{K}_{M}\Gamma^{M}_{H\mu}L_{N}^{-1H} - \frac{\partial L^{K}_{M}}{\partial x^{\mu}} L_{N}^{-1M} \tag{17}$$

$$A'_{\mu} = A_{\mu} \tag{18}$$

where

$$\eta_{KM}L^{K}_{N}L^{M}_{H} = f\eta_{NH} \tag{19}$$

Therefore, $[\Gamma^{K}_{N\mu}]$ transforms as an abstract connection ruled by the Lorentz ($f = 1$) or Weyl group, whereas $[e^{K}_{\mu}]$, $[f_{K\mu}]$ obey homogeneous transformation rules, formally identical with local rotations of tetrads.

We are dealing here with three kinds of linear spaces, $T_xM$, $\mathbb{C}^4$, and $V \simeq \mathbb{R}^4$, and three kinds of indices, $\mu$, $r$, and $K$. It is convenient to unify the corresponding covariant differentiations based on quantities $\{^{\lambda}_{\mu\nu}\}$, $\omega^{r}_{s\mu}$, and $\Gamma^{K}_{L\mu}$ into a single operation, for brevity denoted by $D_{\mu}$. It is clear that

$$D_{\mu}G_{r\bar{}} = 0, \qquad D_{\mu}\gamma^{Ar}_{s} = 0 \tag{20}$$

If $\det[e^{A}_{\mu} \neq 0]$ and $\det[f_{A\mu}] \neq 0$, then, after the $GL(2, \mathbb{C})$ reduction, they become "tetrads" and enable one to construct the following affine connections:

$$\Gamma(e)^{\lambda}_{\mu\nu} := e^{\lambda}_{A}\Gamma^{A}_{B\nu}e^{B}_{\mu} + e^{\lambda}_{A}e^{A}_{\mu,\nu}$$
$$\Gamma(f)^{\lambda}_{\mu\nu} := -f_{A\mu}\Gamma^{A}_{B\nu}f^{\lambda B} + f^{\lambda A}f_{A\mu,\nu} \tag{21}$$

They have their own torsions and curvatures, $S(e)$, $S(f)$, $R(e)$, $R(f)$. Obviously, $\Gamma(e)$ and $\Gamma(f)$ are metrical, respectively, with respect to the following "metric" tensors:

$$h(e)_{\mu\nu} = \eta_{AB}e^{A}_{\mu}e^{B}_{\nu}, \qquad h(f)_{\mu\nu} = \eta^{AB}f_{A\mu}f_{B\nu} \tag{22}$$

Let us introduce the following convenient symbols:

$$E^{A}_{\mu} = \frac{1}{2}(e^{A}_{\mu} + f^{A}_{\mu}), \qquad F^{A}_{\mu} = \frac{1}{2}(e^{A}_{\mu} - f^{A}_{\mu}) \tag{23}$$
$$E^{\mu}_{A} = g^{\mu\nu}\eta_{AB}E^{B}_{\nu}, \qquad F^{\mu}_{A} = g^{\mu\nu}\eta_{AB}F^{B}_{\nu}$$

Our wave equation for $\psi$ may be now rewritten in the form

$$i\gamma^{A}\mathcal{L}_{E_A}\psi + i\,{}^{A}\gamma\mathcal{L}_{F_A}\psi - W\psi + \frac{1}{2g}g^{\mu\nu}D_{\mu}D_{\nu}\psi = 0 \tag{24}$$

where the "covariant Lie derivatives" are defined as

$$\mathcal{L}_{E_A}\psi = E^{\mu}_{A}D_{\mu}\psi + \frac{1}{2}(D_{\mu}E^{\mu}_{A})\psi \tag{25}$$

$$\mathcal{L}_{F_A}\psi = F^{\mu}_{A}D_{\mu}\psi + \frac{1}{2}(D_{\mu}F^{\mu}_{A})\psi$$

and

$$W = \frac{g}{2}\eta_{AB}E^{A}_{\mu}E^{B}_{\nu}g^{\mu\nu}I - \frac{g}{2}\eta_{AB}F^{A}_{\mu}F^{B}_{\nu}g^{\mu\nu} - \frac{c}{2gb}\mathrm{I} \tag{26}$$
$$+ igg^{\mu\nu}E^{A}_{\mu}F^{B}_{\nu}\varepsilon_{ABCD}\Sigma^{CD}$$

## 3. CORRESPONDENCE WITH DIRAC THEORY

The above form of wave equation is "Dirac-like", except for two things:

(i) Two kinds of Dirac matrices are present, corresponding to two normal-hyperbolic signatures $(+---)$, $(-+++)$.

(ii) The second-order d'Alembert term is present.

The first is not very bad, perhaps just desirable if we treat seriously the conformal symmetry. It turns out that (ii) also need not be catastrophic.

One can show that our field equations possess matter-free ($\psi = 0$) solutions with

$$f_{A\mu} = \eta_{AB}e^{B}_{\mu}, \qquad g_{\mu\nu} = h(e)_{\mu\nu} = \eta_{AB}e^{A}_{\mu}e^{B}_{\nu} \tag{27}$$

The geometrodynamical sector, in normal conditions, is weakly sensitive to the matter state, thus, the weak field $\psi$ imposed onto the above background satisfies with a very good accuracy

$$ie^{\mu}_{A}(D_{\mu} + S^{\kappa}{}_{\kappa\mu})\psi - \frac{4bg^2 - c}{2bg}\psi + \frac{1}{2g}g^{\mu\nu}D_{\mu}D_{\nu}\psi = 0 \tag{28}$$

where now $e^{\mu}_{A} = E^{\mu}_{A}$, $F^{\mu}_{A} = 0$, and $S$ is the torsion of $\Gamma(e)$. Without the d'Alembert term, this would be exactly the Dirac equation in Einstein–Cartan space. The special-relativistic approximation when $e^{\mu}_{A} = \delta^{\mu}_{A}$, $S = 0$, and $\Gamma(e) = 0$ has the form:

$$i\gamma^{\mu}\partial_{\mu}\psi - \frac{4bg^2 - c}{2bg}\psi + \frac{1}{2g}\partial^{\mu}\partial_{\mu}\psi = 0 \tag{29}$$

The viability of our model depends essentially on whether the above Dirac–Klein–Gordon equation (DKG) admits a reasonable physical interpretation.

Of course, it does not fit the Bargmann–Wigner classification of Poincaré-irreducible equations, but this need not exclude its physical utility. Let us consider the general case of such a DKG equation with non-specified constants $A$, $B$, $C$:

$$Ai\gamma^{\mu}\partial_{\mu}\psi - B\psi - C\partial^{\mu}\partial_{\mu}\psi = 0 \tag{30}$$

It is easily seen that the general solution is a superposition of two Dirac waves, with masses

$$m^2 = \frac{1}{2C^2}\left(2BC + A^2 \pm \sqrt{A^4 + 4A^2BC}\right) \tag{31}$$

It turns out that instabilities, tachyons, etc., are eliminated if

$$A^2 + 4BC \geq 0 \tag{32}$$

Of course, we are seriously challenged by the mass-doubling. A few answers are *a priori* possible:

(i) Perhaps the splitting is still below the accuracy threshold of our experiments.

(ii) Perhaps, conversely, the gap is so large that we are unable to excite the higher state. Let us recall that the two-particle electron–positron state in the usual Dirac theory may be created only when the frequency of exciting radiation satisfies

$$\nu \geq 2mc^2/h$$

If $C \to 0$ then $m_- \to |B/A|$, $m_+ \to \infty$, and the latter resembles what one is dealing with in Pauli–Villars–Rayski regularization.

(iii) The standard model of electroweak interactions is based on the experimentally evident kinship between massive leptons and their neutrions (also, quarks interact weakly in doublets). If $B = 0$, there is one massive and one massless state in the DKG equation. Perhaps the mass splitting is desirable? Perhaps two massive states of fundamental fermions must be created by the same field, just as the electron and positron are?

(iv) There is no mass splitting at all and the DKG equation reduces to the Dirac case if parameters are so tuned that $A^2 + 4BC = 0$. Then $m_- = m_+ = |B/C|$.

In our special case of the DKG equation, $A = 1$, $B = 2g - c/(2gb)$, $C = -1/(2g)$, and

$$m^2 = \frac{c}{b} - 2g^2\left(1 \pm \sqrt{\frac{c}{bg^2} - 3}\right) \tag{33}$$

The Dirac-like range occurs above the threshold $c/b \geq 3g^2$ at which there is only one mass $m = |g|$. In general, the mass splitting is given by

$$\Delta(m^2) = 4|g| \sqrt{\frac{c}{b} - 3g^2} \tag{34}$$

If $c/b = 4g^2$, one of the masses vanishes, and the other equals $2|g|$. If $c/b = 0$ (and in general, below the threshold $3g^2$), there is no Dirac correspondence, but it is not excluded that the symmetry $GL(4,\mathbb{C})$ could be compatible with fermionic mass generation from nothing (more rigorously, from interactions with the gauge field). To obtain a $GL(4,\mathbb{C})$-invariant theory of bispinors, we must admit the mass form $G_{\bar{\tau}s}$ to be dynamical.

The above model is intimately connected with certain geometrodynamical theory with degrees of freedom $A^r_{s\mu}$, $g_{\mu\nu}$. The very generation of Dirac behavior from the original Klein–Gordon background is a byproduct of the spontaneous symmetry breaking from $U(2, 2)$ to $SL(2, \mathbb{C})$ [or even $GL(2, \mathbb{C})$]. There is no place here for the geometrodynamical sector analysis. Some results concerning this aspect may be found in Sławianowski (1996, 1997).

Here we restrict ourselves to stating that there is a correspondence with Poincaré-gauge theories of gravitation and with Einstein's theory (Hehl *et al.*, 1980; Ponomariov *et al.*, 1985; Sławianowski, 1996, 1997). Substituting in our equations the Einsteinian Ansatz $f_{A\mu} = \eta_{AB}e^B_{\ \mu}$, $g_{\mu\nu} = ph(e)_{\mu\nu}$, $Q_\mu = 0$, $A'_\mu = 0$, $S^\lambda_{\ \mu\nu} = 0$, we reduce them (in the matter-free case, $\psi = 0$), to

$$R(g)_{\mu\nu} - \frac{1}{2} R(g) g_{\mu\nu} = -\frac{12g^2}{p} g_{\mu\nu}$$

$$\left(l - \frac{24g^2 d}{p}\right) g_{\mu\nu} = T_{YM}{}^{\mu\nu}$$

In this way, even if $l = 0$, $d = 0$, i.e., if there is no dynamical Hilbert–Einstein term, we obtain Einstein equations with a cosmological term [controlled by the coupling constant of the $U(2, 2)$-gauge field] from the very Yang–Mills equations for $A$. There are interesting vacuum solutions corresponding to the pure gauge, when $F = 0$. They correspond to the constant-$g$-curvature space.

$$R(g)_{\alpha\beta\mu\nu} = \frac{4g^2}{p} (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu})$$

with the additional condition $lp = 24g^2 d$ satisfied automatically if there is no $L_{HE}$ term in the dynamics.

In a sense, we have obtained the Dirac equation from the Klein–Gordon equation; first differential order from the second one. But one can obtain an answer to our objections by using first-order equations from the very beginning. The model will be more complicated, because its geometrodynamical sector consists of the $U(2, 2)$ connection and the $iu(2, 2)$-valued differential form generalizing the tetrad field. It will be reported in a forthcoming paper.

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